

Restrictions of eigenfunctions to totally geodesic submanifolds of codimension 2

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ABSTRACT. In this paper, we measure the maximum concentration of eigenfunctions restricted to totally geodesic submanifolds of codimension 2. In [1], N. Burq, P. Gérard, and N. Tzvetkov obtained L^p estimates for eigenfunctions restricted to submanifolds. Their results are sharp except a log loss at one endpoint in codimension 1 and 2 submanifolds respectively. The log loss in codimension one can be removed as showed in [2]. Here we remove the log loss of the codimension two case for geodesic submanifolds.

1. Introduction

Suppose (M, g) is a closed Riemannian manifold. Consider the eigenfunctions of the Laplace-Beltrami operator:

$$-\Delta_g \phi_j = \lambda_j^2 \phi_j, \quad \phi_j \in C^\infty(\mathcal{M}), \quad \int_{\mathcal{M}} \phi_j \phi_k dV_g = \delta_{jk}.$$

Here, the eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3, \dots$, are placed in ascending order counted with multiplicity.

One of the main topics regarding eigenfunctions is to measure their concentration. There are several common ways to do this. One way is to consider the growth of the L^p norm of eigenfunctions. A second way is to measure its growth of the L^p norm over some local domains, specifically, geodesic balls or tubes along geodesics. A third way is to consider the growth of L^p norm of eigenfunctions restricted to submanifolds. See [3], [4], [5] etc. for partial references. From a Quantum Physics view, localizations of Quantum States are also important phenomenon. For example, the Anderson Localization helps explain the disorder-induced metal-insulator transition. See [6], [7]. In this paper, we improve an endpoint case of the maximum concentration of eigenfunctions restricted to totally geodesic submanifolds.

Before we state the main result, we will first review some previous results. In [1], N. Burq, P. Gérard, and N. Tzvetkov obtained the following L^p estimates for eigenfunctions restricted to submanifolds:

THEOREM 1. (*N. Burq, P. Gérard, and N. Tzvetkov [1]*)

Let (M, g) be a compact smooth Riemannian manifold of dimension n , and let Σ be a smooth submanifold of dimension k . There exists a constant $C > 0$ such that for any φ_λ , we have

$$\|\varphi_\lambda\|_{L^p(\Sigma)} \leq C(1 + \lambda)^{\rho(k,n)} \|\varphi_\lambda\|_{L^2(M)}$$

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where

$$(1.1) \quad \rho(n-1, n) = \begin{cases} \frac{n-1}{2} - \frac{n-1}{p} & \text{if } p_0 = \frac{2n}{n-1} < p \leq +\infty \\ \frac{n-1}{4} - \frac{n-2}{2p} & \text{if } 2 \leq p < p_0 = \frac{2n}{n-1} \end{cases}$$

$$(1.2) \quad \rho(n-2, n) = \frac{n-1}{2} - \frac{n-2}{p} \quad \text{if } 2 < p \leq +\infty$$

$$(1.3) \quad \rho(k, n) = \frac{n-1}{2} - \frac{k}{p} \quad \text{if } 1 \leq p \leq n-3.$$

If $p = p_0 = \frac{2n}{n-1}$ and $k = n-1$, we have

$$\|\varphi_\lambda\|_{L^p(\Sigma)} \leq C(1 + \lambda)^{\frac{n-1}{2n}} \log^{\frac{1}{2}}(\lambda) \|\varphi_\lambda\|_{L^2(M)}$$

and if $p=2$ and $k=n-2$, we have

$$\|\varphi_\lambda\|_{L^p(\Sigma)} \leq C(1 + \lambda)^{\frac{1}{2}} \log^{\frac{1}{2}}(\lambda) \|\varphi_\lambda\|_{L^2(M)}.$$

These estimates are sharp in the general case except for $(n, k, p) = (n, n-1, \frac{2n}{n-1})$ and $(n, k, p) = (n, n-2, 2)$, which have log loss. Later on, R. Hu gave another proof in [2] of these estimates and showed the log loss for the case $(n, k, p) = (n, n-1, \frac{2n}{n-1})$ can be removed. For the remaining case, in [8] X. Chen and C. Sogge showed that if $n = 3$ and the submanifold is a geodesic, then the log loss can be removed.

In this paper, we deal with the case of $(n, k, p) = (n, n-2, 2)$, where $n \geq 3$ and the submanifold is totally geodesic.

The following is our main result:

THEOREM 2. (Main Theorem)

Let (M, g) be a compact smooth Riemannian manifold of dimension n , and let Σ be a smooth totally geodesic submanifold of dimension $n-2$. There exists a constant $C > 0$ such that for any φ_λ , we have

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \leq C(1 + \lambda)^{\frac{1}{2}} \|\varphi_\lambda\|_{L^2(M)}$$

Sketch of proof: $n = 3$ was proved in [8]. Here we assume $n \geq 4$. First we apply the TT^* argument and reduce the problem to an operator norm bound over the submanifold Σ . Then we expand the kernel of this operator by Hadamard parametrix. For the main term, we do a scaling and compare it to a projection operator with uniform bound over $L^2(\Sigma)$. For all other terms, due to the gains on the exponent, we can use Lemma 2 to control their operator norms.

2. Preliminaries

In this section, we will review and prepare some general results needed in the proof of Main Theorem. First, we need the Hadamard Parametrix, see [9] for references.

LEMMA 1. (Hadamard Parametrix) Let (M, g) be a compact manifold without boundary. If $t \leq \rho$, $\rho > 0$ and ρ is smaller than the injective radius of (M, g) . If $N > n + 3$, then we have:

$$(2.1) \quad (\cos t \sqrt{-\Delta_g})(x; y) = K_N(t, x; y) + R_N(t, x; y)$$

where $R_N \in C^{N-n-3}([-\rho, \rho] \times M \times M)$, and

$$(2.2) \quad K_N(t, x; y) = \begin{cases} \partial_t \left(\sum_{\nu=1}^N \omega_\nu(x, y) E_\nu(t, \kappa(x, y)) \right) & \text{if } t \geq 0 \\ -\partial_t \left(\sum_{\nu=1}^N \omega_\nu(x, y) E_\nu(-t, \kappa(x, y)) \right) & \text{if } t < 0 \end{cases}$$

Here $\kappa(x, y)$ is the vector from x to y in the local geodesic coordinates at x . And $\omega_\nu \in C^\infty(M \times M)$, specially $\omega_0(x, x) = 1, \forall x \in M$.

E_ν are distributions such that

$$(2.3) \quad \partial_t E_0(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \cos(t|\xi|) d\xi,$$

and $E_\nu, \nu = 1, 2, 3, \dots$ is a finite linear combination of Fourier integrals of the form:

$$(2.4) \quad H(t) t^j (2\pi)^{-n} \int_{\mathbb{R}^n \setminus B_1(0)} e^{ix \cdot \xi \pm it|\xi|} |\xi|^{-\nu-1-k} d\xi + \eta_{j\nu}$$

where $j + k = \nu$ and $\eta_{j\nu}$ are smooth.

Further more, we have $\partial_t E_\nu = \frac{t}{2} E_{\nu-1}$.

REMARK 1. We will also use the property that $\omega_0(x, x) = 1, \forall x \in M$ in our proof.

We also need the following lemma in [1](Prop 6.3), which will be used several times in the proof.

LEMMA 2. (N. Burq, P. Gérard, and N. Tzvetkov [1]) Let (N, h) be a compact Riemannian manifold, $\dim N = k$. Q_λ is an operator with kernel

$$(2.5) \quad Q_\lambda(x, y) = \sum_{\pm} \frac{e^{\pm i\lambda d_h(x, y)}}{(\lambda d_h(x, y))^m} a^\pm(x, y, \lambda), \lambda d_h(x, y) > 1$$

and $|Q_\lambda(x, y)| \leq C$. Here $a^\pm(x, y, \lambda) \in C^\infty(N \times N \times \mathbb{R})$, $\partial_{x,y}^\alpha a^\pm \leq C_\alpha$. Then

$$(2.6) \quad \begin{aligned} \|Q_\lambda\|_{L^2(N) \rightarrow L^2(N)} &\lesssim \lambda^{-m - \frac{k-1}{2}} \sum_{j \leq \log \lambda} 2^{j(m - \frac{k+1}{2})} \\ &\lesssim \begin{cases} \lambda^{-k} & \text{if } m > \frac{k+1}{2} \\ \lambda^{-m - \frac{k-1}{2}} & \text{if } m < \frac{k+1}{2} \\ \lambda^{-m - \frac{k-1}{2}} \log \lambda & \text{if } m = \frac{k+1}{2} \end{cases} \end{aligned}$$

Through out this paper, the notation $A \lesssim B$ and $A \gtrsim B$ denote $A \leq CB$ and $A \geq CB$ respectively, for some generic constant C which does not depend on λ .

3. Proof

Without loss of generality, we assume the injective radius of (M, g) is great than 10. Choose any $\chi \in \mathcal{S}(\mathbb{R})$, such that $\chi(0) = 1$, $\text{Supp } \hat{\chi} \subset [1, 2]$. Let $\chi_\lambda f = \chi(\lambda - \sqrt{-\Delta_g})f$, then $\chi_\lambda \varphi_\lambda = \varphi_\lambda$. Thus it suffices to show

$$(3.1) \quad \|\chi_\lambda\|_{L^2(M) \rightarrow L^2(\Sigma)} \lesssim \lambda^{\frac{1}{2}}.$$

By TT^* argument, 3.1 is equivalent to

$$(3.2) \quad \|\chi_\lambda \chi_\lambda^*\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \lesssim \lambda.$$

Let $T_\lambda = \chi_\lambda \chi_\lambda^*$, a simple calculation shows the kernel of $\chi_\lambda \chi_\lambda^*$ is the same as

$$(3.3) \quad \chi^2(\lambda - \sqrt{-\Delta_g})(x, y)|_{\Sigma \times \Sigma}$$

Let $\phi = \chi^2$, then $\phi(0) = 1$, $Supp \hat{\chi} \subset [2, 4]$.

$$(3.4) \quad \begin{aligned} T_\lambda &= \phi(\lambda - \sqrt{-\Delta_g}) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}(t) e^{i(\lambda - \sqrt{-\Delta_g})t} \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \hat{\phi}(t) e^{-i\lambda t} \cos(t\sqrt{-\Delta_g}) - \phi(\lambda + \sqrt{-\Delta_g}). \end{aligned}$$

By Hadamard parametrix, as in Lemma 1, the kernel of T_λ is

$$(3.5) \quad \begin{aligned} T_\lambda(x, y) &= \frac{\omega_0(x, y)}{\pi} \int_{\mathbb{R}} \hat{\phi}(t) e^{i\lambda t} \cdot (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\kappa(x, y) \cdot \xi} \cos(t|\xi|) d\xi dt \\ &\quad + \int_{\mathbb{R}} \hat{\phi}(t) e^{i\lambda t} \sum_{\nu=1}^N \sum_{j=0}^{\nu} a_{\nu j}^{\pm} \omega_\nu(x, y) \partial_t(H(t)t^j (2\pi)^{-n} \int_{\mathbb{R}^n \setminus B_1(0)} e^{ix \cdot \xi \pm it|\xi|} |\xi|^{-2\nu-1+j} d\xi) \\ &\quad + R_N(x, y, \lambda) - \phi(\sqrt{-\Delta_g} + \lambda)(x, y) \\ (3.6) \quad &= \omega_0(x, y) \int_{\mathbb{R}^n} \phi(\lambda - |\xi|) e^{i\kappa(x, y) \cdot \xi} d\xi \\ &\quad + \sum_{\nu=1}^N \sum_{j=1}^{\nu} \omega_\nu(x, y) \int_{\mathbb{R}^n \setminus B_1(0)} \phi_{1j\nu}^{\pm}(\lambda \pm |\xi|) e^{i\kappa(x, y) \cdot \xi} |\xi|^{-2\nu-1+j} d\xi \\ &\quad + \sum_{\nu=1}^N \sum_{j=1}^{\nu} \omega_\nu(x, y) \int_{\mathbb{R}^n \setminus B_1(0)} \phi_{2j\nu}^{\pm}(\lambda \pm |\xi|) e^{i\kappa(x, y) \cdot \xi} |\xi|^{-2\nu+j} d\xi \\ &\quad + \omega_0(x, y) \int_{\mathbb{R}^n} \phi(\lambda + |\xi|) e^{i\kappa(x, y) \cdot \xi} d\xi + \tilde{R}_N(x, y, \lambda) - \phi(\sqrt{-\Delta_g} + \lambda)(x, y). \end{aligned}$$

For the last equality, we used the fact that ϕ is supported in the positive axis. Here $\phi_{1j\nu}^{\pm}$ is the inverse Fourier transform of $(2\pi)^{-n+1} a_{j\nu}^{\pm} \cdot t^{j-1} \hat{\phi}(t)$, and $\phi_{2j\nu}^{\pm}$ is the inverse Fourier transform of $(2\pi)^{-n+1} a_{j\nu}^{\pm} t^j \hat{\phi}(t) \cdot (\pm i)$, which are also Schwartz functions independent with λ .

Next we introduce a new operator which will play an important role in the proof and help us simplify the calculations.

Define S_r^ν , $\nu = 0, 1, 2, 3, \dots$ to be the operator with kernel:

$$(3.7) \quad S_r^\nu(x, y) = \omega_\nu(x, y) \int_{S^{n-1}(1)} e^{ir\kappa(x, y) \cdot \omega} d\omega.$$

By Stationary Phase, see [9] or [3], we can see that S_r^ν satisfies the condition in Lemma 2 with $k = n - 2$ and $m = \frac{n-1}{2}$, thus by Lemma 2, we have the following estimate:

$$(3.8) \quad \|S_r^\nu\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \lesssim \lambda^{-n+2} \log \lambda.$$

Now let us go back to (3.6). There are 5 terms in it and the last three terms can be easily controlled. For the first two terms, by using the spherical coordinates for the ξ

variables, we can rewrite them as

$$\begin{aligned}
 T_\lambda(x, y) &= \int_0^\infty \phi(\lambda - r) S_r^0(x, y) r^{n-1} dr \\
 &\quad + \sum_{\nu=1}^N \sum_{j=1}^\nu \int_1^\infty \phi_{1j\nu}^\pm(\lambda \pm r) S_r^\nu(x, y) r^{-2\nu-1+j} \cdot r^{n-1} dr \\
 &\quad + \sum_{\nu=1}^N \sum_{j=1}^\nu \int_1^\infty \phi_{2j\nu}^\pm(\lambda \pm r) S_r^\nu(x, y) r^{-2\nu+j} \cdot r^{n-1} dr \\
 &= A_\lambda + B_\lambda.
 \end{aligned}$$

Here A_λ is the first term, B_λ is the rest. By using the estimate 3.8, we are able control B_λ as following:

$$\begin{aligned}
 \|B_\lambda\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} &\lesssim \sum_{\nu=1}^N \sum_{j=1}^\nu \int_1^\infty (|\phi_{1j\nu}^\pm(\lambda \pm r)| + |\phi_{2j\nu}^\pm(\lambda \pm r)|) \|S_r^\nu\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} r^{-1} \cdot r^{n-1} dr \\
 &\lesssim \sum_{\nu=1}^N \sum_{j=1}^\nu \int_1^\infty (|\phi_{1j\nu}^\pm(\lambda \pm r)| + |\phi_{2j\nu}^\pm(\lambda \pm r)|) r^{-n+2} \log r \cdot r^{n-2} dr \\
 (3.9) \quad &\lesssim \log \lambda
 \end{aligned}$$

The same procedure will be used several times in the later context.

Similarly, if we can show the following stronger estimate without the log loss for S_r^0 , then we are able to control A_λ as needed, and the proof would be complete. Fortunately, it is true by the fact that $\omega_0 = 1$ on the diagonal:

LEMMA 3. *we have*

$$(3.10) \quad \|S_r^0\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \lesssim r^{-n+2}.$$

PROOF. Since this estimate is a local estimate, without loss of generality, we can assume that Σ is closed. Let $h = g|_\Sigma$, then (Σ, h) is a closed Riemanian manifold.

Denote $\tilde{\kappa}(x, y) : \Sigma \times \Sigma \rightarrow \mathbb{R}^{n-2}$, as the vector from x to y in the local geodesic coordinates with respect to (Σ, h) at x . Similarly, in the later context, any functions or operators under ' ' will be on the submanifold Σ . Since Σ is totally geodesic, we can assume $\kappa|_{\Sigma \times \Sigma} = (\tilde{\kappa}, 0, 0)$. Accordingly, we can make the following change of coordinates:

$$(3.11) \quad B^{n-2}(1) \times [0, 2\pi) \rightarrow S^{n-1} : (z, \sqrt{1-|z|^2} \cos \theta, \sqrt{1-|z|^2} \sin \theta)$$

The Jacobian is 1, thus we can modify the kernel of operator S_r as

$$\begin{aligned}
 S_r(x, y) &= \omega_0(x, y) \int_{B^{n-2}(1)} \int_0^{2\pi} e^{ir\kappa(x, y) \cdot \omega(z, \theta)} d\theta dz \\
 &= 2\pi \omega_0(x, y) \int_{B^{n-2}(1)} e^{ir\tilde{\kappa}(x, y) \cdot z} dz \\
 (3.12) \quad &= 2\pi \omega_0(x, y) r^{-n+2} \int_{B^{n-2}(r)} e^{i\tilde{\kappa}(x, y) \cdot z} dz.
 \end{aligned}$$

Let \overline{S}_r be the operator with kernel

$$(3.13) \quad \overline{S}_r(x, y) = \omega_0(x, y) \int_{B^{n-2}(r)} e^{i\tilde{\kappa}(x, y) \cdot z} dz.$$

Then 3.10 is equivalent to

$$(3.14) \quad \|\overline{S}_r\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \lesssim 1.$$

To prove this estimate, we compare \overline{S}_r to an operator with uniform bound over $L^2(\Sigma)$. Consider the eigenfunctions and eigenvalues of $-\Delta_h$ over Σ :

$$(3.15) \quad -\Delta_h e_{\mu_j} = \mu_j e_{\mu_j}, 0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots.$$

Define P_μ to be the projection map to the eigenspace with eigenvalue $\leq \mu$, that is

$$(3.16) \quad P_\mu = \sum_{\mu_j \leq \mu} E_{\mu_j} = \chi_{[-\mu, \mu]}(\sqrt{-\Delta_g}).$$

Obviously, $\|P_\mu\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \leq 1$. The kernel of P_μ is given by

$$(3.17) \quad \begin{aligned} P_\mu(x, y) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\sqrt{-\Delta_h}} \hat{\chi}_{[-\mu, \mu]}(t) dt \\ &= \frac{1}{\pi} \int_{\mathbb{R}} e^{it\sqrt{-\Delta_h}} \frac{\sin \mu t}{t} dt \\ &= \frac{1}{\pi} \int_{\mathbb{R}} e^{it\sqrt{-\Delta_h}} \beta(t) \frac{\sin \mu t}{t} dt + \frac{1}{\pi} \int_{\mathbb{R}} e^{it\sqrt{-\Delta_h}} (1 - \beta(t)) \frac{\sin \mu t}{t} dt. \end{aligned}$$

Here we choose $\beta(t)$ to be an even cut off function supported in $[-\delta, \delta]$, $\beta(t) = 1$ in $[-\frac{\delta}{2}, \frac{\delta}{2}]$, $\delta > 0$ is less than the injective radius of (Σ, h) . Let r_μ be the inverse Fourier transform of $(1 - \beta(t)) \frac{2\sin \mu t}{t}$, as in [3], r_μ satisfies

$$(3.18) \quad |r_\mu(t)| \leq C_N (1 + ||t| - \mu|)^{-N}, \mu \geq 1, N = 1, 2, 3, \dots.$$

Hence we can rewrite 3.17 as

$$(3.19) \quad P_\mu(x, y) = \frac{1}{\pi} \int_{\mathbb{R}} \beta(t) \frac{\sin \mu t}{t} \cos t \sqrt{-\Delta_h} dt + r_\mu(\sqrt{-\Delta_h}).$$

For the second term, it is a multiplier uniformly bounded over $L^2(\Sigma)$. For the first term, we can compute it by the Hadamard parametrix about (Σ, h) :

$$\begin{aligned}
 P_\mu(x, y) &= \frac{\tilde{\omega}_0(x, y)}{\pi} \int_{\mathbb{R}} (2\pi)^{-n} \int_{\mathbb{R}^{n-2}} \beta(t) \frac{\sin \mu t}{t} e^{i\tilde{\kappa}(x, y) \cdot z} \cos t|z| dt dz \\
 &\quad + \int_{\mathbb{R}} \beta(t) \frac{\sin \mu t}{t} \partial_t \left(\sum_{\nu=1}^N \tilde{\omega}_\nu(x, y) E_\nu(t, \tilde{\kappa}(x, y)) \right) \\
 &\quad + \tilde{R}_N(x, y, \mu) \\
 &= \frac{\tilde{\omega}_0(x, y)}{\pi} \int_{\mathbb{R}} (2\pi)^{-n} \int_{\mathbb{R}^{n-2}} \beta(t) \frac{\sin \mu t}{t} e^{i\tilde{\kappa}(x, y) \cdot z} e^{it|z|} dt dz \\
 &\quad + \int_{\mathbb{R}} \beta(t) \frac{\sin \mu t}{t} \partial_t \left(\sum_{\nu=1}^N \tilde{\omega}_\nu(x, y) E_\nu(t, \tilde{\kappa}(x, y)) \right) \\
 &\quad + \tilde{R}_N(x, y, \mu) \\
 &= \tilde{\omega}_0(x, y) (2\pi)^{-n} \int_{\mathbb{R}^{n-2}} \chi_{[-\mu, \mu]} e^{i\tilde{\kappa}(x, y) \cdot z} dz \\
 &\quad + \frac{\tilde{\omega}_0(x, y)}{\pi} \int_{\mathbb{R}} (2\pi)^{-n} \int_{\mathbb{R}^{n-2}} (1 - \beta(t)) \frac{\sin \mu t}{t} e^{i\tilde{\kappa}(x, y) \cdot z} e^{it|z|} dt dz \\
 &\quad + \int_{\mathbb{R}} \beta(t) \frac{\sin \mu t}{t} \partial_t \left(\sum_{\nu=1}^N \tilde{\omega}_\nu(x, y) E_\nu(t, \tilde{\kappa}(x, y)) \right) \\
 &\quad + \tilde{R}_N(x, y, \mu) \\
 (3.20) \quad &= \tilde{A}_\mu + \tilde{B}_\mu + \tilde{C}_\mu + \tilde{D}_\mu.
 \end{aligned}$$

We have included the $r_\mu(\sqrt{-\Delta_h})$ into the remainder term $\tilde{R}_N(x, y, \mu)$ here. The uniform boundedness of this remainder term \tilde{D}_μ over $L^2(\Sigma)$ is trivial when $N > n + 3$.

For \tilde{B}_μ , using spherical coordinates, we know that

$$(3.21) \quad \tilde{B}_\mu = (2\pi)^{-n} \int_1^\infty r_\mu(\rho) \rho^{n-3} \tilde{S}_\rho^\nu(x, y) d\rho$$

where

$$(3.22) \quad \tilde{S}_\rho^\nu(x, y) = \tilde{\omega}_\nu(x, y) \int_{S^{n-3}(1)} e^{ir\tilde{\kappa}(x, y) \cdot z} dz.$$

Then by Stationary Phase, we can see that S_r^ν satisfies the condition in 2 with $k = n - 2$ and $m = \frac{n-3}{2}$. Thus by Lemma 2, we have the following estimate:

$$(3.23) \quad \|\tilde{S}_\rho^\nu\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \lesssim \rho^{-n+3}.$$

So

$$(3.24) \quad \|B_\mu\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \lesssim \int_0^\infty (1 + |\mu - \rho|)^{-N} \rho^{n-3} \cdot \rho^{-\frac{n-3}{2} - \frac{n-2-1}{2}} d\rho \lesssim 1.$$

For \tilde{C}_μ , notice that for each $E_\nu, \nu = 1, 2, 3, \dots$, is a finite linear combination of Fourier integrals of the form:

$$(3.25) \quad H(t) t^j (2\pi)^{-n+2} \int_{\mathbb{R}^{n-2} \setminus B_1(0)} e^{ix \cdot z \pm it|z|} |z|^{-2\nu-1+j} dz + \eta_{j\nu}, j = 0, 1, 2, \dots, \nu,$$

where $\eta_{j\nu}$ are smooth functions, thus their contributions can be ignored. When $\nu = 1$, a detailed calculation (see [9]) shows that

$$(3.26) \quad E_1(t, x) = \frac{H(t)}{2} \times (2\pi)^{-n} \int_{R^{n-2} \setminus B_1(0)} e^{ix \cdot z} \left(\frac{\sin t|z|}{|z|} - t \cos t|z| \right) \frac{dz}{|z|^2}.$$

Again, since $\beta(t)$ is even, we can simplify the corresponding term as

$$(3.27) \quad \tilde{C}_\mu^1 = c_{1\mu} \int_{\mathbb{R}} \int_1^\infty \beta(t) \sin \mu t \sin \rho t \tilde{S}_\rho^\nu \rho^{n-4} d\rho dt.$$

So

$$(3.28) \quad \|\tilde{C}_\mu^1\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \lesssim \sum_{\pm} \int_1^\infty |\beta(\mu \pm \rho)| \rho^{-n+3} \cdot \rho^{n-4} d\rho \lesssim \frac{1}{\mu}.$$

We can deal with $\nu \geq 2$ similarly by $\partial_t E_\nu = \frac{t}{2} E_{\nu-1}$.

From above, we know that \tilde{A}_μ is uniformly bounded on $L^2(\Sigma)$, so is $\overline{P}_\mu = (2\pi)^n \tilde{A}_\mu$. Let $\mu = r$ and consider the difference between \overline{S}_r and \overline{P}_r :

$$(3.29) \quad \overline{S}_r(x, y) - \overline{P}_r(x, y) = (\omega_0(x, y) - \tilde{\omega}_0(x, y)) \int_{B^{n-2}(r)} e^{i\tilde{\kappa}(x, y) \cdot z} dz.$$

Since $\omega_0(x, x) = \tilde{\omega}_0(x, x) = 1$, we have $\omega_0(x, y) - \tilde{\omega}_0(x, y) = O(d_h(x, y))$, thus by Stationary Phase and Lemma 2, we know

$$(3.30) \quad \|\overline{S}_r - \overline{P}_r\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \lesssim 1.$$

Finally we conclude that

$$(3.31) \quad \|\overline{S}_r\|_{L^2(\Sigma) \rightarrow L^2(\Sigma)} \lesssim 1,$$

and this completes the proof. □

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